

# IRREDUCIBLE POLYNOMIALS WHICH ARE LOCALLY REDUCIBLE EVERYWHERE

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ABSTRACT. For any positive integer  $n$ , there exist polynomials  $f(x) \in \mathbb{Z}[x]$  of degree  $n$  which are irreducible over  $\mathbb{Q}$  and reducible over  $\mathbb{Q}_p$  for all primes  $p$ , if and only if  $n$  is composite. In fact, this result holds over arbitrary global fields.

## 1. INTRODUCTION

Hilbert gave examples of irreducible polynomials  $f(x) \in \mathbb{Z}[x]$  of degree 4 which are reducible mod  $p$  for all primes  $p$ , namely  $x^4 + 2ax^2 + b^2$ . Note that this polynomial is irreducible over  $\mathbb{Q}(a, b)$  hence (by Hilbert's irreducibility theorem) is irreducible over  $\mathbb{Q}$  for infinitely many specializations of  $a$  and  $b$  into  $\mathbb{Q}$ . The underlying reason for this phenomenon from the Galois theoretic point of view is that the Galois group of  $x^4 + 2ax^2 + b^2$  over  $\mathbb{Q}(a, b)$  is Klein's four group. Therefore for any  $p$  not dividing the discriminant of  $f$ , the decomposition group is a cyclic group of order at most 2, so  $f$  is reducible mod  $p$ . (Note that for  $p$  dividing the discriminant of  $f$ ,  $f$  is reducible mod  $p$  as well.) The phenomenon is thus forced by the structure of the Galois group. This also explains why there can be no such examples of polynomials of prime degree. Indeed, suppose  $f(x) \in \mathbb{Z}[x]$  has prime degree  $\ell$  and is irreducible in  $\mathbb{Z}[x]$ . Then its Galois group has an element of order  $\ell$ , so by Chebotarev's density theorem there exists  $p$  such that the splitting field of  $f$  over  $\mathbb{F}_p$  has Galois group  $C_\ell$ , the cyclic group of order  $\ell$ , hence  $f$  must be irreducible over  $\mathbb{F}_p$ . We will show that the degree of  $f$  being prime is the only obstacle, namely that for any composite  $n$ , there exist irreducible  $f(x) \in \mathbb{Z}[x]$  of degree  $n$  which are reducible mod  $p$  for all  $p$ . In fact, there is an irreducible  $f(t, x) \in \mathbb{Z}[t, x]$  of degree  $n$  such that  $f(t_0, x)$  is reducible mod  $p$  for all specializations  $t = t_0$  in  $\mathbb{Z}$  and all  $p$ . We will also prove that for any composite  $n$ , there exist irreducible  $f(x) \in \mathbb{Q}[x]$  of degree  $n$  which are reducible over  $\mathbb{Q}_p$  for all  $p$ , and that this result generalizes to arbitrary global fields. Note that Hilbert's example does not satisfy this last condition for all  $a, b$ , e.g.  $x^4 + 1$  is irreducible over  $\mathbb{Q}_2$ .

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It is worthwhile pointing out here that a random polynomial  $f(x) \in \mathbb{Z}$  of composite degree  $n$  is not reducible mod  $p$  for all  $p$ , as its Galois group over  $\mathbb{Q}$  is  $S_n$  [5], and since  $S_n$  contains an  $n$ -cycle, Chebotarev's density theorem implies that there are infinitely many primes  $p$  for which  $f(x)$  is irreducible mod  $p$ .

## 2. MOD $p$ REDUCIBILITY

Let  $f(x) \in \mathbb{Z}[x]$  be monic irreducible of degree  $n$  with Galois group  $G$  over  $\mathbb{Q}$ . As in the discussion above, we see that if  $f(x)$  is irreducible mod  $p$ , then  $p$  does not divide the discriminant of  $f(x)$ , and  $G$  must contain an element of order  $n$ , since the decomposition group is cyclic of order  $n$ . Thus if  $G$  has no element of order  $n$ , then  $f$  must be reducible mod  $p$  for all  $p$ . On the other hand, since  $f$  is irreducible over  $\mathbb{Z}$ ,  $G$  has a subgroup  $H$  of index  $n$ . More generally, if  $K/\mathbb{Q}$  is a finite Galois extension with Galois group  $G$ , and if  $G$  has a subgroup  $H$  of index  $n$  but no element of order  $n$ , then the fixed field, say  $\mathbb{Q}(\theta)$ , of  $H$  has degree  $n$  and the minimal polynomial  $f$  of  $\theta$  over  $\mathbb{Q}$  has degree  $n$  with splitting field  $L \subseteq K$ , Galois group  $\bar{G} = G(L/\mathbb{Q}) \cong G/\text{core}(H)$  where  $\text{core}(H)$  is the intersection of the conjugates of  $H$ . Furthermore,  $\bar{G}$  has a subgroup  $\bar{H}$  of index  $n$  but has no element of order  $n$ , so  $f$  is reducible mod  $p$  for every  $p$ . In summary:

**Lemma 2.1.** *Let  $G$  be a finite group and  $n$  a positive integer such that*

- 1)  *$G$  is realizable as a Galois group  $G(K/\mathbb{Q})$ ,*
- 2)  *$G$  has a subgroup of index  $n$ , but  $G$  has no element of order  $n$ .*

*Then there is an irreducible polynomial  $f(x) \in \mathbb{Z}[x]$  of degree  $n$  which is reducible mod  $p$  for all primes  $p$  (with splitting field contained in  $K$ ).*

In connection with Hilbert's example, it follows from the preceding considerations that a polynomial of degree four has the property of being irreducible over  $\mathbb{Z}$  and reducible mod  $p$  for all  $p$  if and only if its Galois group over  $\mathbb{Q}$  is either Klein's four group  $V_4$  or the alternating group  $A_4$ .

**Theorem 2.2.** *For any composite positive integer  $n$ , there exist irreducible polynomials  $f(x) \in \mathbb{Z}[x]$  of degree  $n$  which are reducible mod  $p$  for all primes  $p$ .*

*Proof.* By the lemma, it suffices to find  $G$  satisfying conditions 1) and 2).

*Case 1.*  $n$  squarefree. Write  $n = qm$  with  $q$  prime, so  $(m, q) = 1$ . Let  $t$  be the order of  $q$  mod  $m$  and set  $G = C_m \rtimes V$  (semidirect product), where  $V$  denotes the additive group of  $\mathbb{F}_{q^t}$  with  $C_m$  acting by multiplication by the  $m$ th roots of unity  $\mu_m \subset \mathbb{F}_{q^t}$ , where we identify  $C_m$  with  $\mu_m$ . The action of  $C_m$  on  $V$  is faithful and 0 is the only fixed point. Let  $H$  be a hyperplane in  $V$ , i.e.  $[V : H] = q$ . Then  $[G : H] = qm = n$ . Furthermore, if  $g \in G$  had order  $n$ , then  $g^m \in V$  would have order  $q$  and be fixed by  $g$  hence by  $C_m$ , contradiction.

We therefore have found for every squarefree composite  $n$ , a solvable group  $G$  satisfying condition 2). By Shafarevich's theorem (in fact by an older theorem of Scholz),  $G$  is realizable as a Galois group over  $\mathbb{Q}$ .

*Case 2.*  $n$  not squarefree. Assume first that  $n = q^2$ ,  $q$  prime. Then  $G_1 := C_q \times C_q$  satisfies 2) with  $H$  the trivial group. For arbitrary  $n = q^2 m$ , take  $G := G_1 \times C_m$  which again has the trivial subgroup of index  $n$  and no element of order  $n$ .  $\square$

We now remark that since the groups  $G$  appearing in the proof can be realized regularly over the rational function field  $\mathbb{Q}(t)$  (see e.g. [3, p. 275]), we obtain irreducible polynomials  $f(t, x) \in \mathbb{Z}[t, x]$  of degree  $n$  which by Hilbert's irreducibility theorem have infinitely many specializations of  $t$  into  $\mathbb{Q}$  which are irreducible with Galois group  $G$ , hence reducible mod  $p$  for all  $p$ .

### 3. $p$ -ADIC REDUCIBILITY

We now wish to prove Theorem 2.2 with reducibility mod  $p$  replaced by reducibility over  $\mathbb{Q}_p$ . The preceding construction actually yields irreducible polynomials  $f(x) \in \mathbb{Z}[x]$  which are reducible over  $\mathbb{Q}_p$  for all primes which are unramified in the splitting field of  $f$ , but may be irreducible over  $\mathbb{Q}_p$  for ramified  $p$ . The proof will be similar but more delicate.

Let  $f(x) \in \mathbb{Q}[x]$  be irreducible and let  $p$  be a prime. Let  $K$  be the splitting field of  $f$  over  $\mathbb{Q}$ ,  $G = G(K/\mathbb{Q})$ ,  $\mathfrak{p}$  a prime of  $K$  over  $p$ ,  $D = D(\mathfrak{p})$  the decomposition group. Then  $f$  is irreducible over  $\mathbb{Q}_p$  if and only if  $D(\mathfrak{p})$  acts transitively on the roots of  $f$  in  $K$ . Let  $H$  be the subgroup of  $G$  fixing a root of  $f$ . The action of  $G$  on the roots of  $f$  is equivalent to its action by multiplication from the left on the left cosets of  $H$  in  $G$ . Thus  $f$  is irreducible over  $\mathbb{Q}_p$  if and only if  $D(\mathfrak{p})$  acts transitively on the left cosets of  $H$  in  $G$ , i.e. the set product  $DH$  is equal to  $G$ . Suppose  $K/\mathbb{Q}$  were tamely ramified, so that all the decomposition groups were metacyclic. Then in order to insure that  $f$  is reducible over all  $\mathbb{Q}_p$ , it would suffice that (the set product)  $MH \neq G$  for every metacyclic subgroup  $M$  of  $G$ . We therefore have the following lemma.

**Lemma 3.1.** *Let  $G$  be a finite group and  $n$  a positive integer such that*

- 3)  *$G$  is realizable over  $\mathbb{Q}$  by a tamely ramified extension  $K/\mathbb{Q}$ ,*
- 4)  *$G$  has a subgroup  $H$  of index  $n$ , the intersection of whose conjugates is trivial, such that the set product  $MH \neq G$  for every metacyclic subgroup  $M$  of  $G$ .*

*Then there exists an irreducible polynomial  $f(x) \in \mathbb{Q}[x]$  of degree  $n$  (with splitting field  $K$ ) which is reducible over  $\mathbb{Q}_p$  for all primes  $p$ .*

**Theorem 3.2.** *For any composite positive integer  $n$ , there exist irreducible polynomials  $f(x) \in \mathbb{Q}[x]$  of degree  $n$  which are reducible over  $\mathbb{Q}_p$  for all primes  $p$ .*

*Proof.* By Lemma 3.1, it suffices to find  $G$  satisfying 3) and 4).

*Case 1.*  $n$  squarefree. The same  $G$  and  $H$  work as in the proof of Theorem 2.2, provided the order  $t$  of  $q$  mod  $m$  is greater than 1 (which can be ensured by taking  $q$  to be the smallest prime dividing  $n$ ). Indeed,  $V$  has  $\mathbb{F}_q$ -dimension  $t$  and is irreducible as a  $C_m$ -module, so in this case the only metacyclic subgroups of  $G$  are cyclic of order  $q$  or a divisor of  $m$ . Thus condition 4) is satisfied.

Condition 3) can be proved using Saltman's results [4, Theorem 3.5] to verify that  $G$  has a generic Galois extension and then [4, Theorem 5.9] to see that there exists a tame extension realizing  $G$ .

*Case 2.  $n$  not squarefree.* As in the proof of Theorem 2.2, we first assume  $n = q^2$ ,  $q$  a prime. We will construct a Galois extension  $K/\mathbb{Q}$  with Galois group  $C_q \times C_q$ , with local degree 1 or  $q$  at all primes  $p$ . If  $K = \mathbb{Q}(\theta)$  with  $f(x)$  the minimal polynomial of  $\theta$  over  $\mathbb{Q}$ , then  $f$  has the desired property. As we saw earlier, for any prime unramified in  $K$  the decomposition group is cyclic, so the local degree is 1 or  $q$  just from the structure of  $G$ . We need to construct  $K$  so that the local degree is  $q$  at the ramified primes. For this we use an idea from [2].

Let  $\ell$  be a prime congruent to 1 mod  $q$  and let  $L_\ell \subseteq \mathbb{Q}(\mu_\ell)$  such that  $G(L_\ell/\mathbb{Q}) \cong C_q$ . We seek another prime  $r$  such that

- 5)  $r \equiv 1 \pmod{q}$
- 6)  $r$  splits completely in  $L_\ell$
- 7)  $\ell$  splits completely in  $L_r$

If  $r$  satisfies these conditions, then clearly  $K = L_\ell L_r$  will have the local degree at most  $q$  everywhere.

5) is equivalent to the condition that  $r$  splits completely in  $\mathbb{Q}(\mu_q)$ . 7) is equivalent to the condition that the Frobenius automorphism of  $\ell$  in  $G(\mathbb{Q}(\mu_r)/\mathbb{Q})$  fixes  $L_r$  pointwise, i.e. is a  $q$ th power in the cyclic group  $G(\mathbb{Q}(\mu_r)/\mathbb{Q})$ , which is equivalent to  $\ell$  being a  $q$ th power mod  $r$ , which in turn means that the polynomial  $x^q - \ell$  has a root mod  $r$ . Since  $\mathbb{F}_r$  contains the  $q$ th roots of unity, this is equivalent to  $x^q - \ell$  factoring into linear factors mod  $r$ , which is equivalent to the condition that  $r$  splits completely in  $\mathbb{Q}(\mu_q, \sqrt[q]{\ell})$ . It follows that conditions 5), 6), 7) together are equivalent to the condition that  $r$  splits completely in  $L_\ell(\mu_q, \sqrt[q]{\ell})$ . By Chebotarev's density theorem, such an  $r$  exists. This completes the case  $n = q^2$ .

Now assume the general case  $n = q^2 m$ ,  $q$  prime. Let  $K$  be as in the case  $n = q^2$ , and let  $L$  be any abelian extension of  $\mathbb{Q}$  of degree  $m$  such that  $K \cap L = \mathbb{Q}$ . (For example, choose a prime  $p \equiv 1 \pmod{m}$ ,  $p \neq \ell, r$ , and let  $L \subseteq \mathbb{Q}(\mu_p)$  of degree  $m$  over  $\mathbb{Q}$ .)  $KL/\mathbb{Q}$  has degree  $n$ , and the local degree at any prime is at most  $qm < n$ .  $\square$

#### 4. GLOBAL FIELDS

Theorem 3.2 generalizes to arbitrary global fields  $F$ . If  $n$  is prime and  $f(x) \in F[x]$  is a separable monic irreducible polynomial, then by Chebotarev's density theorem, which holds over any global field [6, p. 289], there exist primes  $\mathfrak{p}$  of  $F$  such that  $f$  is irreducible over the completion  $F_{\mathfrak{p}}$ . (Even if  $n$  is equal to the characteristic  $p$  of  $F$ , and  $f(x)$  is inseparable of degree  $p$ , i.e.  $f(x) = x^p - a$ , then by [1, Chapter 9, Theorem 1], if  $f(x)$  is irreducible, then  $f(x)$  is irreducible over  $F_{\mathfrak{p}}$  for infinitely many  $\mathfrak{p}$ .)

**Theorem 4.1.** *For any composite positive integer  $n$ , and any global field  $F$ , there exist irreducible polynomials  $f(x) \in F[x]$  of degree  $n$  which are reducible over  $F_{\mathfrak{p}}$  for all primes  $\mathfrak{p}$  of  $F$ .*

*Proof.* If  $F$  is a number field, one can reduce the proof to the case  $F = \mathbb{Q}$ . Let  $f(x) \in \mathbb{Q}[x]$  be irreducible of degree  $n$  and reducible over  $\mathbb{Q}_p$  for all  $p$ , and suppose its splitting field  $K$  satisfies  $K \cap F = \mathbb{Q}$ . Then  $f(x)$  is irreducible over  $F$  and is reducible

over  $F_{\mathfrak{p}}$  for all primes  $\mathfrak{p}$  of  $F$ . It remains to observe that the proof of Theorem 3.2 produces infinitely many  $\mathbb{Q}$ -linearly disjoint extensions  $K$  with the desired properties.

We may therefore assume that  $F$  a global function field of characteristic  $p$ .

Let  $f(x) \in F[x]$  be monic irreducible of degree  $n$ , fix a root  $\alpha$  in a splitting field  $K$ , let  $G = G(K/F)$ , and let  $H = G(K/F(\alpha))$ . Let  $\mathfrak{p}$  be a prime of  $F$ ,  $\mathfrak{P}$  a prime of  $K$  dividing  $\mathfrak{p}$ ,  $D = D(\mathfrak{P})$  the decomposition group. Then  $f(x)$  is reducible over  $F_{\mathfrak{p}}$  if and only if  $D$  does not act transitively on the roots of  $f(x)$ , i.e. the set  $DH$  is not equal to  $G$ . We reduce the proof to the case  $n$  a product of two primes, not necessarily distinct.

For any positive integer  $m$ , let  $E/F$  be a Galois (e.g. cyclic) extension of degree  $m$  such that  $K \cap E = F$ , with  $K$  as above. Assume for all primes  $\mathfrak{P}$  of  $K$  that  $D(\mathfrak{P})H \neq G$ . Let  $\hat{K} = KE$ ,  $\hat{G} = G(KE/F) \cong G \times G(E/F)$ . Identify  $H$  with  $H \times \{1\} \subset \hat{G}$ . Then for any prime  $\hat{\mathfrak{P}}$  of  $\hat{K}$ ,  $D(\hat{\mathfrak{P}})H \neq \hat{G}$ . This reduces the proof of Theorem 4.1 to the case  $n$  a product of two primes, say  $r$  and  $s$ .

We will further reduce the proof to the case  $F$  is a rational function field  $\mathbb{F}_q(t)$ , by constructing the desired  $K/\mathbb{F}_q(t)$  linearly disjoint over  $\mathbb{F}_q(t)$  from any  $F$  given in advance. Accordingly we now assume  $F = \mathbb{F}_q(t)$ . As we will see, when  $n$  is prime to  $p$ , one can give a proof which is analogous to that of Theorem 3.2.

*Case 1.*  $r = s \neq p$ . We use a function field analogue of a construction in [2]. Let  $\mathfrak{p} = (h(t))$  be a (finite) prime of  $F$  ( $h(t) \in \mathbb{F}_q[t]$ ) which splits completely in the extension  $F'$  of  $F$  obtained by adjoining all  $r$ th roots of all elements of  $\mathbb{F}_q$  (including the  $r$ th roots of unity). Then  $Cl_F(\mathfrak{p})$ , the ray class group mod  $\mathfrak{p}$ , is cyclic (the class group of  $F$  is trivial), and has order divisible by  $r$ . There is a corresponding ray class field extension  $R^{\mathfrak{p}}/F$  which is geometric (regular over  $\mathbb{F}_q$ ),  $G(R^{\mathfrak{p}}/F) \cong Cl_F(\mathfrak{p})$ , and  $\mathfrak{p}$  is the only prime that ramifies in  $R^{\mathfrak{p}}$ . Let  $L^{\mathfrak{p}}$  be the (unique, cyclic) subfield of  $R^{\mathfrak{p}}$  of degree  $r$  over  $F$ . We seek  $\mathfrak{p}, \mathfrak{q}$  such that  $K := L^{\mathfrak{p}}L^{\mathfrak{q}}$  has the desired property, which is that  $\mathfrak{p}$  splits completely in  $L^{\mathfrak{q}}$  and  $\mathfrak{q}$  splits completely in  $L^{\mathfrak{p}}$ . Let  $\mathfrak{p}$  be as above, which exists e.g. by Chebotarev's density theorem (we don't really need Chebotarev yet since the extension is a constant extension, but presently we will need it). We seek  $\mathfrak{q}$  which satisfies the same conditions as does  $\mathfrak{p}$ , and additionally,  $\mathfrak{p}$  splits completely in  $L^{\mathfrak{q}}$  and  $\mathfrak{q}$  splits completely in  $L^{\mathfrak{p}}$ . The last condition is an additional Chebotarev condition on  $\mathfrak{q}$  which is compatible with the preceding ones, so can be satisfied by Chebotarev's density theorem. The condition  $\mathfrak{p}$  splits completely in  $L^{\mathfrak{q}}$  is equivalent to the condition that the Frobenius  $Frob(\mathfrak{p})$  is an  $r$ th power in  $G(R^{\mathfrak{q}}/F)$ , which is equivalent to  $\mathfrak{p} = h(t)$  being an  $r$ th power in  $Cl_F(\mathfrak{q})$ , which is equivalent to  $h(t)$  being an  $r$ th power in the multiplicative group of the residue field  $\mathbb{F}_q[t]/\mathfrak{q}$ , which is equivalent to  $\mathfrak{q}$  splitting completely in  $F(\sqrt[r]{h(t)})$ . This is another Chebotarev condition on  $\mathfrak{q}$  compatible with the preceding ones. The construction yields infinitely many linearly disjoint extensions  $K$ , completing the proof of this case.

*Case 2.*  $r \neq s$ , both  $\neq p$ . Here we can argue exactly as in the case  $n$  squarefree in the proof of Theorem 3.2, by constructing a tamely ramified  $G$ -extension using Saltman's results as quoted earlier. This yields infinitely many linearly disjoint extensions.

*Case 3.*  $r = s = p$ . We seek a pair of Artin-Schreier extensions  $L, M$  defined by

$x^p - x - g(t)$ ,  $x^p - x - h(t)$  respectively, where  $g(t), h(t) \in F$ , such that  $K := LM$  has local degree 1 or  $p$  everywhere. Let  $g(t)$  be a polynomial without constant term, i.e.  $g(0) = 0$ . Since the derivative of  $x^p - x - g(t)$  with respect to  $x$  is  $-1$ , the only prime of  $F$  that ramifies in  $L$  is infinity. Furthermore, the prime corresponding to  $t$  splits completely in  $L$  since  $x^p - x - g(t)$  has  $p$  distinct roots mod  $t$ . The automorphism  $t \mapsto 1/t$  interchanges  $(t)$  and infinity. Let  $h(t) = g(1/t)$ . Then infinity splits completely in  $M$  and  $(t)$  is the only ramified prime in  $M$ . It follows that the local degree of  $LM$  is 1 or  $p$  everywhere, as desired. We get infinitely many linearly disjoint extensions by varying  $g(t)$ , e.g. by taking  $g(t)$  of the form  $t^e$ , with  $e$  prime to  $p$ . The genus of the corresponding curve grows with  $e$ , or one can show that there are infinitely many such distinct extensions even over the completion of  $F$  at infinity.

*Case 4.  $r = p \neq s$ .*

**Subcase 4.1.**  $p \nmid s - 1$ . We can use the group  $G$  we used earlier. Let  $C_p$  act on an irreducible  $\mathbb{F}_s$ -space  $V$ , so  $\dim(V) > 1$ . Claim that  $G := C_p \ltimes V$  cannot be a local Galois group anywhere. Indeed, if it were, the extension would be ramified, so the inertia group would be a nontrivial normal subgroup of  $G$ , hence equal to  $V$  or  $G$ . There is no wild ramification, since  $G$  has no normal  $p$ -subgroup. But then the inertia subgroup must be cyclic, contradicting the fact that  $G$  has no cyclic normal subgroup. This proves the claim. It follows that all decomposition groups are proper subgroups of  $G$ . But if  $D$  is a proper subgroup of  $G$ , then  $DH$  cannot equal  $G$ , as is easily verified (see beginning of the previous section). It remains to show that  $G$  is realizable (infinitely often, linearly disjointly) over  $F$ . Realize  $C_p$  by an Artin-Schreier extension  $L/F$  (there are infinitely many), and consider the embedding problem with kernel  $V$ . It has a proper solution since  $F$  is Hilbertian [3, p. 275].

**Subcase 4.2.**  $p \mid s - 1$ .

Here the roles of  $p$  and  $s$  need to be interchanged, and we have to worry about wild ramification. (The argument will also cover subcase 4.1.) Let  $E/F$  be a cyclic regular extension of degree  $s$ . Claim there exists an Artin-Schreier extension  $L/E$  of degree  $p$  such that every prime  $\mathfrak{q}$  of  $E$  that ramifies in  $L$  is split completely over  $F$  and all its remaining  $s - 1$  conjugates over  $F$  are unramified in  $L$ . Indeed, there are infinitely many primes  $\mathfrak{p}$  of  $F$  that split completely in  $E$ . Let  $\{\mathfrak{p}_i\}$  be a sequence of such primes and let  $\mathfrak{q}_i$  be a prime of  $E$  dividing  $\mathfrak{p}_i$  for each  $i$ . Let  $h = h_E$  be the class number of  $E$ . Then in the sequence  $\mathfrak{q}_1, \mathfrak{q}_1\mathfrak{q}_2, \dots, \mathfrak{q}_1\mathfrak{q}_2 \cdots \mathfrak{q}_{h+1}$  of ideals of  $E$ , there must be two which differ by a principal ideal  $(f)$ ; i.e. there exist  $i \leq j \in \{1, 2, \dots, h + 1\}$  such that  $\mathfrak{q}_i \cdots \mathfrak{q}_j = (f)$  with  $f \in E^*$ . Take  $L/E$  to be the Artin-Schreier extension defined by the equation  $x^p - x = 1/f$ . Since the only poles of  $1/f$  are  $\mathfrak{q}_i, \dots, \mathfrak{q}_j$ , the only ramified primes in  $L/E$  are also  $\mathfrak{q}_i, \dots, \mathfrak{q}_j$ . Furthermore, since these are simple poles, the extension  $L/E$  is nontrivial (of degree  $p$ ). This proves the claim.

Let  $M/F$  be the Galois closure of  $L/F$ , and set  $W = G(M/E)$ . By construction, the conjugates of  $L$  over  $F$  are linearly disjoint over  $E$ , so  $W$  is isomorphic to the group ring  $\mathbb{F}_p C_s$  as  $C_s$ -modules, identifying  $G(E/F)$  with  $C_s$ . There is an irreducible submodule  $V$  of  $W$  on which  $C_s$  acts faithfully. The  $\mathbb{F}_p$ -dimension of  $V$  is necessarily greater than 1 since  $s > p$ . Let  $K$  be the subfield of  $M$  corresponding to the complementary submodule to  $V$  in  $W$  (by Maschke's theorem). Then  $K/F$  is Galois with group  $G = C_s \ltimes V$ . Let  $H$  be a subgroup of  $V$  of index  $p$ . We show that  $H$  has

the desired property, that  $DH \neq G$  for any decomposition group  $D$ . The argument is similar to previous ones. If  $D$  is the decomposition group of an unramified prime, then  $D$  is cyclic. Since  $G$  has no cyclic subgroup of order  $sp$ ,  $D$  has order  $p$  or  $s$ , hence  $DH$  cannot equal  $G$ . At a tamely ramified prime, the inertia group must be of order  $s$  and be normal in  $D$ . So  $D$  is either cyclic of order  $s$ , and  $DH \neq G$ , or  $D$  is of order  $sp$  with a normal  $p$ -Sylow subgroup, which cannot happen because  $D$  has no element of order  $sp$ . Finally, the decomposition group of a wildly ramified prime in  $K/F$  must be contained in  $V$ , since such a prime must divide one of  $\mathfrak{q}_i, \dots, \mathfrak{q}_j$ , and  $C_s$  does not fix any of them. Hence again  $DH \neq G$ . Finally, there are infinitely many linearly disjoint extensions  $E/F$ , hence infinitely many linearly disjoint extensions  $K/F$ . The proof is complete.  $\square$

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